The Olson conjecture for discrete public goods

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Abstract

We consider the private provision of a public good with non-refundable binary contributions. A fixed amount of the good is provided if and only if the number of contributors reaches an exogenous threshold. The threshold, the group size, and the identical cost of contributing to the public good are common knowledge. Our main result shows that the maximal probability of reaching the threshold (and thereby obtaining the public good) which can be supported in a symmetric equilibrium of this participation game is decreasing in group size. This generalizes a well-known result for the volunteer's dilemma – in which the threshold is one – to arbitrary thresholds and thereby confirms a conjecture by Olson for the class of participation games under consideration. Further results characterize the limit when group size goes to infinity and provide conditions under which the expected number of contributors is decreasing or increasing in group size.

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1. Introduction

Ever since Olson (1965), group size has been considered an important determinant of the likelihood that a group will be successful in attaining its common goals. While Olson's maxim that "[t]he larger a group is, the farther it will fall short of providing the optimal supply of any collective good, and the less likely that it will act to obtain even a minimal amount of such a good" (Olson, 1965, p.36) has attracted much attention in economics and political science, providing firm theoretical underpinnings for it has proven challenging (Sandler, 2015). While the conditions under which Olson's maxim is valid are well understood for some of the standard models of collective action (e.g., Chamberlin, 1974; McGuire, 1974; Andreoni, 1988), for other such models this is not the case.

In this paper we are concerned with the second part of Olson's maxim, namely the statement that "[t]he larger the group is, [...] the less likely that it will act to obtain even a minimal amount of such a good." We refer to this as *the Olson conjecture* and investigate its theoretical foundations for the class of participation games without refunds introduced in Palfrey and Rosenthal (1984) to model the private provision of a discrete public good.[1] These games have inspired much subsequent theoretical work not only in economics and political science (e.g., Palfrey and Rosenthal, 1988; Gradstein and Nitzan, 1990; Dixit and Olson, 2000; Goeree and Holt, 2005; McBride, 2006; Myatt and Wallace, 2008; Makris, 2009), but also in biology (e.g., Bach et al., 2006; Archetti, 2009; Souza et al., 2009; Archetti and Scheuring, 2011; Peña et al., 2014), making them a natural choice for such an investigation.[2] Indeed, we find it surprising that very few attempts (that we discuss below) have been made to verify or disprove the Olson conjecture in this benchmark model for the private provision of a private good.

In the class of participation games we consider, group members decide simultaneously whether to contribute to a public good or not (to participate or not). All contributors pay a non-refundable cost. The public good is provided if and only if the number of contributors reaches an exogenous threshold. If the threshold is passed, all group members receive the same benefit from the provision of the public good (that we normalize to one). The threshold, the group size, and the identical cost of contributing to the public good are known to all players.

The simplest version of such a participation game is the volunteer's dilemma (Diekmann, 1985), in which one contributor suffices to ensure the provision of the public good. For this game the unique symmetric (Nash) equilibrium is straightforward to calculate. It is equally straightforward to show that the probability that

[1] Palfrey and Rosenthal (1984) also consider participation games with refunds. Section 4 discusses the extension of our analysis to this model.

[2] For surveys of experimental evidence relating to such participation games see, for instance, Palfrey and Rosenthal (1988), Ledyard (1995), and Schram et al. (2008).

the public good is provided in this equilibrium is a decreasing function of group size (see, for instance, the textbook treatment in Dixit et al. 2004, p. 454–458). In other words, it is well known that the Olson conjecture holds for the symmetric equilibria of the volunteer's dilemma.

Our main result (Proposition 4 in Section 3.5) shows that the validity of the Olson conjecture does not hinge on the assumption that the threshold for the provision of the public good is $k = 1$ (as in the volunteer's dilemma) but obtains for arbitrary thresholds $k > 1$, that is, for the whole class of participation games without refunds from Palfrey and Rosenthal (1984). More precisely, we show that the maximal probability of reaching the threshold (and thereby obtaining the public good) which can be supported in a symmetric equilibrium is decreasing in group size.[3]

Validating the Olson conjecture for thresholds $k > 1$ is much more subtle than doing so for the volunteer's dilemma. First, the indifference condition for a non-trivial equilibrium (requiring the probability that a player is pivotal for the provision of the public good to equal the participation cost) can no longer be solved explicitly for the probability that players participate.[4] Second, and more importantly, the underlying logic is more intricate. In the volunteer's dilemma the probability of being pivotal is the same as the probability that the contributions of the other group members are insufficient to obtain the public good. Hence, the pivotality condition implies that in equilibrium the probability that the contributions of the other group members are sufficient to obtain the public good is independent of group size. Together with the intuitive result that the equilibrium probability of contributing to the public good is decreasing in group size, this directly implies the Olson conjecture for the volunteer's dilemma.[5] With $k > 1$ this logic breaks down because the probability of being pivotal no longer equals the probability that the contributions of the other group members are insufficient to obtain the public good. Therefore, observing that the probability of contributing to the public good is decreasing in group size (which is still easy to show, cf. Proposition 2 in Section 3.3) does not suffice to imply the Olson conjecture. What is needed is an additional argument, showing that the probability that the contributions of the other group members suffice to obtain the public good is either constant (as in the volunteer's dilemma) or decreasing in group size. Our main technical contribution is to use the pivotality condition to establish that for $k > 1$ this probability, which coincides with the equilibrium payoff, is in fact decreasing in group size (Proposition 3 in Section 3.4).

We complement our investigation of the Olson conjecture by characterizing the

[3] We motivate our focus on such "maximal" symmetric equilibria in Section 2.

[4] Our formal analysis in Section 3 accounts for the possibility that the only symmetric equilibrium of the participation game is trivial in the sense that all group members choose not to participate and the public good is not provided.

[5] Anderson and Engers (2007, Section 2) provide a particularly lucid exposition of this logic. See also Section 3.1 below.

limit when group size goes to infinity and by investigating the comparative statics of the expected number of contributors. These two issues are closely related because (i) the limit results are a fairly straightforward consequence of the observation that – as one would expect – the distribution of the number of contributors converges to a Poisson distribution and (ii) this convergence result allows us to characterize the effect of group size on the expected number of contributors for large groups. Our key finding concerning the second point is that for sufficiently large groups there is a critical cost level such that for costs below this level the expected number of contributors is increasing in group size, whereas for costs above this level (but sufficiently low as to ensure that the probability of provision stays positive; cf. footnote 4) the expected number of contributors is decreasing in group size. With the expected number of contributors being a natural measure of how much agents contribute to the provision of the public good, we find the observation that the expected number of contributors can be increasing in group size (over a fairly large part of the relevant parameter space) interesting because it indicates that the validity of the Olson conjecture is not driven by total contributions being decreasing in group size.

The validity of the Olson conjecture for the participation games from Palfrey and Rosenthal (1984) with thresholds $k > 1$ has been investigated previously in Hindriks and Pancs (2002).[6] In their Proposition 6 these authors observe first, as we do in our Proposition 2, that the probability of contribution is decreasing in group size. The second part of the same proposition claims that "the effect of group size on the probability of provision is indeterminate." In contrast, our main result (Proposition 4) shows that this effect can be determined and is negative for the maximal equilibrium we consider. Hindriks and Pancs (2002, Proposition 7) also consider the case of large group sizes. While three out of the four claims in that proposition are (as we prove) correct, their argument yielding these results is not. The problem is that Hindriks and Pancs (2002) take for granted that the error introduced by using the Poisson approximation to the binomial probability distribution can be ignored when studying comparative statics for sufficiently large groups. Our result on the expected number of contributors – which directly contradicts the corresponding claim in Proposition 7 from Hindriks and Pancs (2002) – shows that this is not so.

Section 2 presents the model and introduces our terminology and notation. Section 3 contains the results described above. Section 4 discusses other, less directly related literature and extensions of our analysis.

[6] The main focus of Hindriks and Pancs (2002) is on the effect of altruism on the equilibrium provision of the public good, but for the case $k > 1$ the version of altruism they consider does not affect the probability that the public good is provided and can therefore be ignored (Hindriks and Pancs, 2002, Proposition 4).

2. Model

2.1. The participation game

The game we consider is the complete-information participation game from Palfrey and Rosenthal (1984) with non-refundable contributions. There is a group of $n > 2$ players. Players simultaneously decide whether or not to participate in the provision of a public good by making a fixed contribution. If k or more group members contribute, the public good is provided. Otherwise, the public good is not provided. Every player derives a benefit, which we normalize to one, if the public good is provided and pays the participation cost c if contributing. Payoffs are the difference between the benefit obtained and the cost incurred. The participation cost satisfies $0 < c < 1$.

For $k = 1$ this participation game is the *volunteer's dilemma* from Diekmann (1985) in which one contributor suffices to ensure the provision of the public good. As the volunteer's dilemma is well understood (see Section 3.1 below), we will focus on the case in which threshold and group size satisfy $1 < k < n$ in the following.[7]

2.2. Symmetric strategy profiles and equilibria

The above participation game has a multitude of asymmetric (Nash) equilibria, including those in which the players coordinate in such a way that k players contribute and the remaining $n - k$ players do not contribute.[8] If such equilibria, in which the public good is provided for sure, are admitted, there are no group-size effects on the probability that the public good is provided, so that the question we are interested in here becomes moot. We therefore focus on the symmetric equilibria of the participation game in which all players contribute with the same probability $p \in [0, 1]$.

We identify any symmetric strategy profile with the common participation probability p and let

$$\pi_{k,n}(p) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \tag{1}$$

denote the probability that any player is pivotal for the provision of the public good when such a strategy profile is played: as the participation of a player will make the difference between provision and non-provision of the public good if and only if $k - 1$ out of the $n - 1$ other players contribute, this probability is given by the binomial expression on the right side of (1).

[7]In the case $k = n$, which is an assurance problem in the sense of Sen (1967), it is an equilibrium for all group members to participate, thereby ensuring the provision of the public good. We exclude this case as it adds nothing of interest to our analysis in Section 3 but would require an additional case distinction.

[8]Further asymmetric equilibria are described in Palfrey and Rosenthal (1984, Section 2).

As we have assumed $1 < k < n$, it is clear that $p = 0$ is a symmetric equilibrium whereas $p = 1$ is not. Further, a symmetric strategy profile with $0 < p < 1$ is a symmetric equilibrium if and only if the players are indifferent between participating or not. This indifference condition requires the probability $\pi_{k,n}(p)$ that a player is pivotal to be equal to the participation cost:

$$\pi_{k,n}(p) = c. \tag{2}$$

It is easily verified that the pivot probability $\pi_{k,n}(p)$ has the following *unimodality properties*: it is differentiable in p, satisfies $\pi_{k,n}(0) = \pi_{k,n}(1) = 0$, is (strictly) increasing on the interval $[0, (k-1)/(n-1)]$ and (strictly) decreasing on the interval $[(k-1)/(n-1), 1]$ with non-zero derivative on the interiors of these intervals.[9] In particular, $(k-1)/(n-1)$ is the unique maximizer of the pivot probability $\pi_{k,n}(p)$ in the interval $[0, 1]$. Hence,

$$\bar{c}_{k,n} = \pi_{k,n}((k-1)/(n-1)) \in (0, 1) \tag{3}$$

is the critical value of the participation cost such that for costs above this level the pivotality condition (2) has no solution and, thus, no interior symmetric equilibrium exists. If $c = \bar{c}_{k,n}$ holds, then $(k-1)/(n-1)$ is the unique solution to (2). If $c < \bar{c}_{k,n}$ holds, the unimodality properties of $\pi_{k,n}(p)$ imply that (2) has one solution to the left and one solution to the right of $(k-1)/(n-1)$. This gives the following characterization result for the number and location of symmetric strategy equilibria, which specializes the characterization results in Palfrey and Rosenthal (1984, Section 2) to the symmetric equilibria under consideration here. Figure 1 illustrates.

Lemma 1. *For $1 < k < n$ the number and location of symmetric equilibria depends on c as follows:*

1. *If $c > \bar{c}_{k,n}$, then $p = 0$ is the unique symmetric equilibrium.*
2. *If $c = \bar{c}_{k,n}$, then there are two symmetric equilibria, namely $p = 0$ and $p = (k-1)/(n-1)$.*
3. *If $c < \bar{c}_{k,n}$, then there are three symmetric equilibria, namely $p = 0$, the unique solution to (2) in the interval $(0, (k-1)/(n-1))$, and the unique solution to (2) in the interval $((k-1)/(n-1), 1)$.*

In the following we will refer to the symmetric equilibrium with $p = 0$ as the *trivial equilibrium* and to symmetric equilibria with $p > 0$ as *non-trivial equilibria*.

[9] The first two of these properties are immediate from the definition of $\pi_{k,n}(p)$ in (2). The remaining properties follow from observing that $\ln(\pi_{k,n}(p)) = (k-1)\ln(p) + (n-k)\ln(1-p) + \ln(\binom{n-1}{k-1})$ is strictly concave in p on $(0, 1)$ and has its unique critical point at $p = (k-1)/(n-1)$.

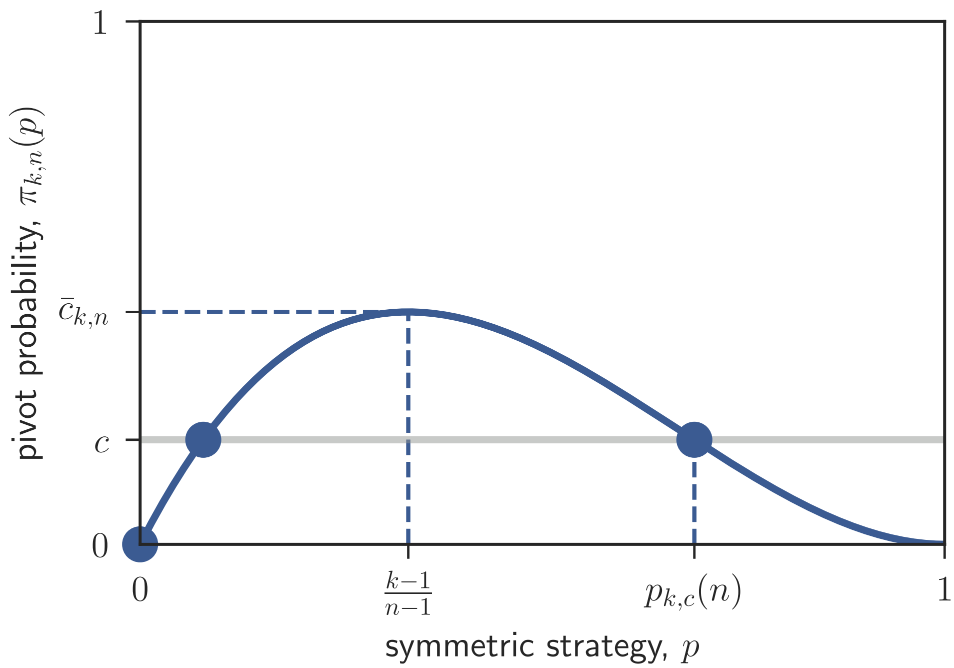

Figure 1: Symmetric equilibria (*circles*) and the pivot probability (*solid line*), here illustrated for $k = 2$, $n = 4$, and $c = 0.2$. The pivot probability $\pi_{k,c}(p)$ is unimodal and has a unique maximum at $(k-1)/(n-1)$ with corresponding pivot probability $\bar{c}_{k,n} \approx 0.44$. For $c < \bar{c}_{k,n}$ there are three symmetric equilibria with the maximal equilibrium, denoted by $p_{k,c}(n)$, satisfying $p_{k,c}(n) > (k-1)/(n-1)$.

2.3. The maximal equilibrium

To deal with the multiplicity of symmetric equilibria arising in cases 2 and 3 of Lemma 1 we impose a further equilibrium selection, focusing on what we refer to as the maximal equilibrium. We first offer formal definitions and then discuss our motivation for this particular selection.

Let $\mathcal{P}_{k,c,n}$ denote the set of symmetric equilibria for given parameter values (k, c, n) and let

$$p_{k,c}(n) = \max \mathcal{P}_{k,c,n} \tag{4}$$

denote the equilibrium with the maximal probability of contribution p among the symmetric equilibria. By Lemma 1, the *maximal equilibrium* $p_{k,c}(n)$ is well defined (because the finite set $\mathcal{P}_{k,c,n}$ is non-empty) and different from the trivial equilibrium $p = 0$ if and only if $c \leq \bar{c}_{k,n}$ holds.

The probability $\phi_{k,c}(n)$ that the public good is provided in the maximal equilibrium is

$$\phi_{k,c}(n) = \Pi_{k,n}(p_{k,c}(n)), \tag{5}$$

where

$$\Pi_{k,n}(p) = \sum_{\ell=k}^{n} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell} \tag{6}$$

is the probability that at least k out of the n group members participate in the symmetric strategy profile p.

The expected payoff $u_{k,c}(n)$ in the maximal equilibrium is

$$u_{k,c}(n) = \phi_{k,c}(n) - p_{k,c}(n) \cdot c = \Pi_{k,n-1}(p_{k,c}(n)). \tag{7}$$

The second equality in (7) holds because the maximal equilibrium satisfies $p_{k,c}(n) < 1$, so that every player obtains the same payoff from choosing not to participate as from following the equilibrium strategy.

For ease of exposition, we will often refer to $p_{k,c}(n)$ simply as the probability of contribution, to $\phi_{k,c}(n)$ as the probability of provision, and to $u_{k,c}(n)$ as the equilibrium payoff. Similarly, we will refer to

$$\mu_{k,c}(n) = n \cdot p_{k,c}(n) \tag{8}$$

as the expected number of contributors.

Our equilibrium selection rule of focusing on the maximal equilibria puts all groups on the same footing by supposing that, no matter how large the group is, all group members contribute with the highest probability that is consistent with a symmetric equilibrium.[10] We demonstrate in Appendix A.1 that the maximal equilibrium also has the appealing property of having the highest probability of provision and the highest expected payoff among the symmetric equilibria. Hence, our subsequent analysis not only reveals how the maximal probability of provision that can be supported in a symmetric equilibrium depends on group size but can also be understood as characterizing the comparative statics of the Pareto-best symmetric equilibrium.

[10] Of course, supposing that the trivial equilibrium is always played also puts all groups on the same footing. However, doing so leads to the trivial conclusion that the public good can never be provided. Any selection rule which selects between the maximal equilibrium and the trivial equilibrium as a function of group size will either lead to results analogous to the ones we derive later or lead to a trivial failure of the Olson conjecture by presupposing that an increase in group size causes the equilibrium to switch from the trivial equilibrium to a maximal equilibrium with a positive probability of provision. Concerning the non-maximal and non-trivial equilibrium that arises in case 3 of Lemma 1, we feel justified in neglecting this equilibrium because it is unstable (McBride, 2006).

3. Results

Section 3.1 briefly reviews the volunteer's dilemma with threshold $k = 1$ to establish a benchmark.

Sections 3.2 – 3.7 consider participation games with thresholds $k > 1$. First, we identify how group size relates to the existence of non-trivial equilibria, with Proposition 1 in Section 3.2 establishing that if non-trivial equilibria fail to exist for some group size, then the same is true for all larger group sizes. Obviously, if there are no non-trivial equilibria, that is, $p = 0$ is the unique symmetric equilibrium, then the probability of contribution, the probability of provision, the equilibrium payoff, and the expected number of contributors are all zero. Thus, in the following subsections we focus our investigation on the range of group sizes for which a non-trivial equilibrium exists.

Section 3.3 shows that over this range of group sizes the probability of contribution is decreasing in group size. Section 3.4 establishes the much deeper result that these comparative statics carry over to the equilibrium payoff. Section 3.5 shows that the Olson conjecture, i.e., the probability of provision is decreasing in group size, is an immediate implication of these two results.

We record limit results for $n \to \infty$ in Section 3.6. In particular, we show that for sufficiently low cost non-trivial equilibria exist for all group sizes and characterize the limiting value of the probability of provision and the equilibrium payoff in terms of the Poisson distribution. Finally, building on these limit results, we consider the comparative statics of the expected number of contributors for large groups in Section 3.7.

3.1. The volunteer's dilemma as a benchmark

Using the terminology and notation introduced in Section 2.3, the probability of contribution, the probability of provision, and the equilibrium payoff in the unique symmetric (and thus maximal) equilibrium for the volunteer's dilemma are given by (cf. Diekmann, 1985)

$$p_{1,c}(n) = 1 - c^{1/(n-1)}, \quad \phi_{1,c}(n) = 1 - c^{n/(n-1)}, \quad u_{1,c}(n) = 1 - c. \tag{9}$$

While it is immediate from the expression for the probability of provision $\phi_{1,c}(n)$ in (9) that the Olson conjecture holds for the volunteer's dilemma, for our subsequent analysis it is more instructive to note that this result can be inferred by observing that the probability of contribution is decreasing in group size whereas the equilibrium payoff is constant in group size. Specifically, from (7) we can write the probability of provision for the volunteer's dilemma as

$$\phi_{1,c}(n) = u_{1,c}(n) + p_{1,c}(n) \cdot c, \tag{10}$$

so that the mere knowledge that $p_{1,c}(n)$ is decreasing in n and that $u_{1,c}(n)$ is independent of n suffices to infer that $\phi_{1,c}(n)$ is decreasing in n. Further, to obtain the result that the equilibrium payoff is independent of group size it suffices to observe the following. First, the equilibrium payoff is, as indicated in (7), equal to the probability that at least one out of $n-1$ group members will participate. Second, because the complementary probability that none of $n-1$ group members participates is the pivot probability and the pivot probability is equal to c in equilibrium, the equilibrium payoff must be $1-c$ for all group sizes. Hence, for the volunteer's dilemma the validity of the Olson conjecture can be understood as a direct consequence of the probability of contribution being decreasing in group size.

While it is immediate from (8) and (9) that the expected number of contributors for the volunteer's dilemma is given by $\mu_{1,c}(n) = n \cdot \left(1 - c^{1/(n-1)}\right)$, the comparative statics of this expression are less obvious and we delay further discussion until Section 3.7.

Using stars to denote limits when group size goes to infinity, we have

$$p^*_{1,c} = 0, \phi^*_{1,c} = u^*_{1,c} = 1 - c, \text{ and } \mu^*_{1,c} = \ln(1/c). \tag{11}$$

The first three of the limit values in (11) are immediate from (9). The limit $\mu^*_{1,c}$ for the expected number of contributors obtains because $p_{1,c}(n)$ converges to zero, so that we can write

$$\lim_{n\to\infty} \mu_{1,c}(n) = \lim_{n\to\infty} n \cdot p_{1,c}(n) = \lim_{n\to\infty} n \cdot p_{1,c}(n+1)$$

and $n \cdot p_{1,c}(n+1) = n \cdot \left(1 - c^{1/n}\right)$ converges (see, for instance, Sydsæter and Hammond 1995, Chapter 8.3) to $-\ln(c)$.

3.2. Group size and the existence of non-trivial equilibria

While the unique symmetric equilibrium for the volunteer's dilemma is non-trivial for all group sizes and participation costs $0 < c < 1$, for participation games with $k > 1$ non-trivial equilibria only exist if the participation cost c does not exceed the critical cost level $\bar{c}_{k,n}$ identified in (3). The following lemma shows that, as one would expect, the critical cost level $\bar{c}_{k,n}$ is decreasing in group size.

Lemma 2. *Let $1 < k < n$. Then $\bar{c}_{k,n+1} < \bar{c}_{k,n}$ holds.*

Proof. Straightforward algebra shows that the equation $\pi_{k,n}(p) = \pi_{k,n+1}(p)$ has a unique solution in the interval $(0, 1)$ given by $\hat{p} = (k-1)/n$. From the unimodality properties of the pivot probability $\pi_{k,n}(p)$ noted in Section 2.2, $(k-1)/(n-1)$ is the unique maximizer of $\pi_{k,n}(p)$ over $p \in (0, 1)$, so that $\pi_{k,n}((k-1)/(n-1)) > \pi_{k,n}((k-1)/n)$ holds. Thus, we have $\pi_{k,n}((k-1)/(n-1)) > \pi_{k,n+1}((k-1)/n)$. Recalling the definition of $\bar{c}_{k,n}$ in (3) we thus have $\bar{c}_{k,n} > \bar{c}_{k,n+1}$. □

Let

$$\bar{n}_{k,c} = \sup_n \{n > k : c \leq \bar{c}_{k,n}\}, \tag{12}$$

where $\bar{n}_{k,c}$ may be infinite and we define it to be equal to k if the supremum is taken over the empty set. From Lemma 1, a non-trivial equilibrium exists if and only if n is in the set appearing on the right side of (12). From Lemma 2, n is in this set if and only if $n \leq \bar{n}_{k,c}$ holds. If this is the case, then the maximal equilibrium is non-trivial and it is immediate from (5) – (8) that the probability of contribution, the expected payoff, and the expected number of contributors are all positive. On the other hand, for $n > \bar{n}_{k,c}$ the maximal equilibrium is trivial, so that the probability of contribution, the expected payoff, and the expected number of contributors are all zero. The following proposition summarizes these observations.

Proposition 1. *Let $1 < k < n$. Then a non-trivial equilibrium exists if and only if $n \leq \bar{n}_{k,c}$. Furthermore,*

$$n \leq \bar{n}_{k,c} \Rightarrow p_{k,c}(n) > 0, \quad \phi_{k,c}(n) > 0, \quad u_{k,c}(n) > 0, \quad \mu_{k,c}(n) > 0, \tag{13}$$

$$n > \bar{n}_{k,c} \Rightarrow p_{k,c}(n) = \phi_{k,c}(n) = u_{k,c}(n) = \mu_{k,c}(n) = 0. \tag{14}$$

We note that Proposition 1 precludes neither the possibility that for given threshold $k > 1$ and participation cost $0 < c < 1$ the trivial equilibrium is the unique symmetric equilibrium for all $n > k$ nor the possibility that for all $n > k$ a non-trivial equilibrium exists. The first of these cases ($\bar{n}_{k,c} = k$) arises if and only if $c > \bar{c}_{k,k+1}$ holds, i.e., the participation cost c is so high that even when the group size is only $k+1$ the probability of being pivotal is lower than c in any symmetric strategy profile. The exact condition under which the second case ($\bar{n}_{k,c} = \infty$) arises is determined in Section 3.6.

Figure 2 illustrates Proposition 1 for a participation game in which $\bar{n}_{k,c} > k$ is finite. It also shows that over the range of group sizes for which a non-trivial equilibrium exists ($k < n \leq \bar{n}_{k,c}$) the probability of contribution, the equilibrium payoff, and the probability of provision are all decreasing in n. Propositions 2 – 4 in the following three subsections establish that these comparative statics obtain for all threshold public good games with $k > 1$.

3.3. The effect of group size on the probability of contribution

The maximal probability of contribution that can be sustained in a symmetric equilibrium is decreasing in group size for $n \leq \bar{n}_{k,c}$ (with Proposition 1 ensuring that it drops to zero thereafter):

Proposition 2. *Let $1 < k < n < \bar{n}_{k,c}$. Then $p_{k,c}(n+1) < p_{k,c}(n)$ holds.*

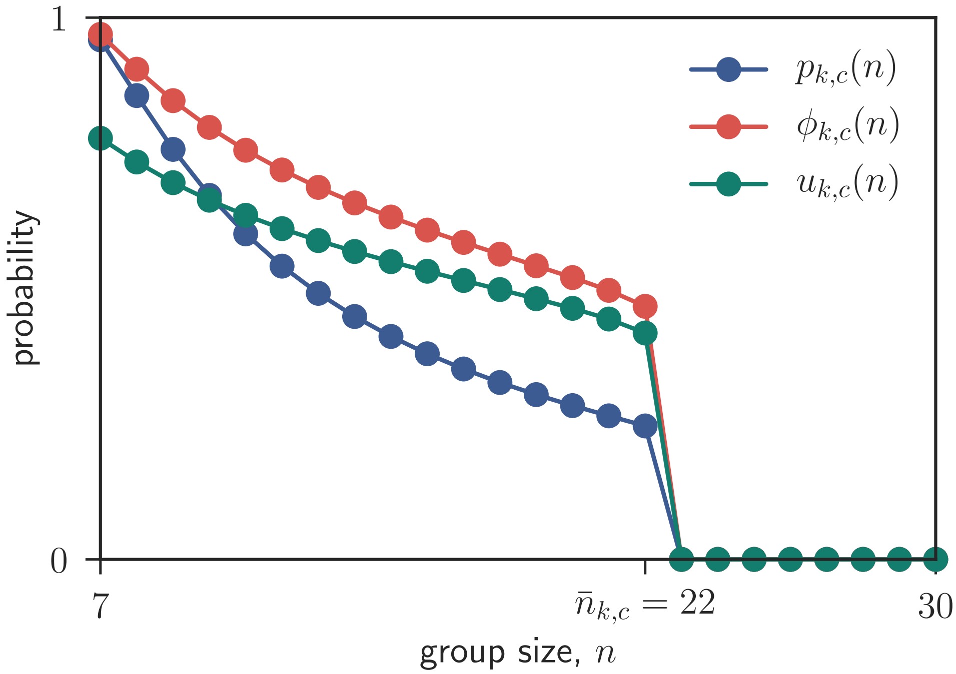

Figure 2: Illustration of Proposition 1 for $k = 6$ and $c = 0.2$. The probability of contribution $p_{k,c}(n)$, the equilibrium payoff $u_{k,c}(n)$, and the probability of provision $\phi_{k,c}(n)$ are all positive up to the critical value $\bar{n}_{k,c} = 22$. For larger groups $p_{k,c}(n)$, $u_{k,c}(n)$, and $\phi_{k,c}(n)$ are all equal to zero. The figure also illustrates Propositions 2 – 4: $p_{k,c}(n)$, $u_{k,c}(n)$, and $\phi_{k,c}(n)$ are all decreasing in group size for $k < n \leq \bar{n}_{k,c}$.

This result has been noted before by Offerman (1997, Theorem 2.3) and Hindriks and Pancs (2002, Proposition 6.i). We provide a proof in Appendix A.2 to make the paper self-contained and to prepare the ground for the proof of Proposition 4. Figure 3 illustrates graphically how Proposition 2 results from the relationship between the pivot probabilities $\pi_{k,n}(p)$ and $\pi_{k,n+1}(p)$.

3.4. The effect of group size on the equilibrium payoff

In any symmetric equilibrium the payoff of every group member is the same as if they were to choose non-participation while the other group members follow the equilibrium strategy. Consequently, as we have noted in equation (7), the equilibrium payoff (in the maximal equilibrium) satisfies

$$u_{k,c}(n) = \Pi_{k,n-1}(p_{k,c}(n)). \tag{15}$$

It is clear from (15) that there are two countervailing effects of an increase in group size on the equilibrium payoff: First, increasing group size for a given probability of contribution p raises the probability that at least k of the $n-1$ other group members

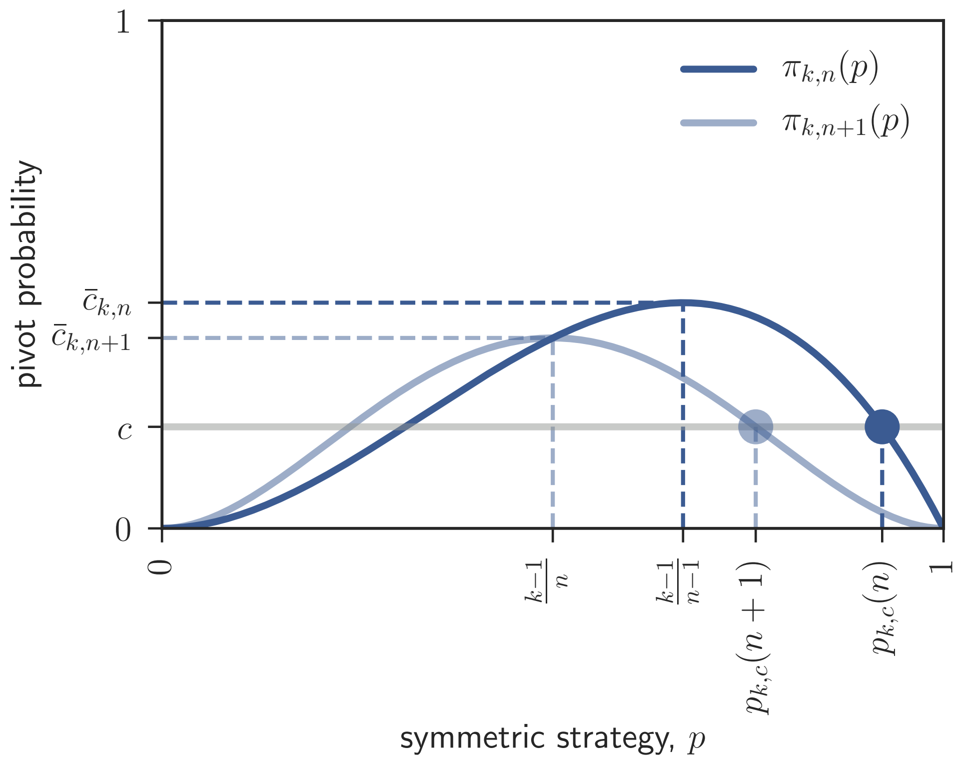

Figure 3: Illustration of Proposition 2 for $k = 3$, $n = 4$, and $c = 0.2$. The pivot probability $\pi_{k,n+1}(p)$ lies below $\pi_{k,n}(p)$ to the right of $(k-1)/n$, implying that the maximal equilibria for consecutive group sizes satisfy $p_{k,c}(n+1) < p_{k,c}(n)$.

provide and therefore has a positive effect on the equilibrium payoff. Second, we know from Proposition 2 that an increase in group size implies a decrease in the probability of contribution, which causes the probability that at least k of the other group members provide to fall and therefore has a negative effect on the equilibrium payoff.

For the volunteer's dilemma these two effects cancel out, so that (as noted in Section 3.1) the equilibrium payoff is independent of group size. For the non-trivial maximal equilibrium in a participation game with $k > 1$ it is no longer the case that the two countervailing effects of an increase in group size on the equilibrium payoff cancel out. Rather, as the proof of the following proposition shows, the net effect is negative: an increase in group size decreases the maximal payoff that can be sustained in a symmetric equilibrium.

Proposition 3. *Let $1 < k < n < \bar{n}_{k,c}$. Then $u_{k,c}(n+1) < u_{k,c}(n)$ holds.*

The proof of Proposition 3 is in Appendix A.3. It proceeds in two steps. The first step shows that the equality $\pi_{k,n+1}(p_{k,c}(n+1)) = \pi_{k,n}(p_{k,c}(n))$, which holds by the pivotality condition (2), implies that the probability that exactly k out of $n-1$ group members participate is decreasing in group size, i.e.,

$$\pi_{k+1,n+1}(p_{k,c}(n+1)) < \pi_{k+1,n}(p_{k,c}(n)) \tag{16}$$

holds. The second step shows how inequality (16) implies that the probability that at most $k-1$ out of $n-1$ group members participate is increasing in group size, so that the complementary probability $\Pi_{k,n-1}(p_{k,c}(n))$ is decreasing in group size. Both steps of the proof rely on a unimodality property of the ratio of binomial probabilities for two different sample sizes from Klenke and Mattner (2010).

It is essential for the result in Proposition 3 that we consider the maximal equilibria for group sizes n and $n+1$. Indeed, arguments analogous to the one proving Proposition 3 show that considering the smaller of the two solutions to the pivotality condition for both group sizes reverses the result: the payoff in these equilibria is *increasing* in group size.

3.5. The effect of group size on the probability of provision

As we have done in Section 3.1 for the volunteer's dilemma, we can rearrange the first equality in (7) to obtain

$$\phi_{k,c}(n) = u_{k,c}(n) + p_{k,c}(n) \cdot c, \tag{17}$$

thereby decomposing the effect of group size on the probability of provision into two effects, namely the effect on equilibrium payoff and the effect on the probability of contribution. From Propositions 2 and 3 both of these effects point in the same direction. Hence, the following proposition, establishing the validity of the Olson conjecture over the range of group sizes for which the probability of provision is positive, is now immediate:

Proposition 4 (Olson conjecture)**.** *Let $1 < k < n < \bar{n}_{k,c}$. Then $\phi_{k,c}(n+1) < \phi_{k,c}(n)$ holds.*

The result in Proposition 4 is in sharp contrast to the corresponding claim in Proposition 6 of Hindriks and Pancs (2002), who assert that "the effect of group size on the probability of provision is indeterminate." There are two sources for this divergence in results. The first is that Hindriks and Pancs (2002) actually do not show that the effect is indeterminate but simply observe that there are two countervailing effects of an increase in group size on the probability of provision. In contrast, our Proposition 4 establishes that for the maximal equilibrium the negative effect of an increase in group size on the probability of contribution dominates the positive effect of having more potential contributors. The second is that Hindriks and Pancs (2002) do not restrict attention to the maximal equilibrium for each group size. As we have noted in Section 3.4, Proposition 3 is not applicable when considering the lower of the two non-trivial equilibria, so that our analysis does not exclude the possibility that the result in Proposition 4 hinges on our equilibrium selection rule.

3.6. Limit results

From the monotonicity results in Sections 3.2 – 3.5 the limits

$$p^*_{k,c} = \lim_{n\to\infty} p_{k,c}(n), \quad \phi^*_{k,c} = \lim_{n\to\infty} \phi_{k,c}(n), \quad u^*_{k,c} = \lim_{n\to\infty} u_{k,c}(n)$$

are well defined. As we will see below, the same is true for the limit of the expected number of contributors,

$$\mu^*_{k,c} = \lim_{n\to\infty} \mu_{k,c}(n).$$

For the case of the volunteer's dilemma ($k = 1$) these limits are given by (11) in Section 3.1. Here we provide the corresponding limit results for $k > 1$.

From the classical Poisson approximation to the binomial distribution and equations (1) and (3) we have

$$\bar{c}^*_k := \lim_{n\to\infty} \bar{c}_{k,n} = g(k-1, k-1), \tag{18}$$

where

$$g(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots, \tag{19}$$

denotes the probability mass function of a Poisson distribution with parameter $\lambda > 0$.

Clearly, $\bar{n}_{k,c}$ is finite if $c > \bar{c}^*_k$ holds. Hence, the following is immediate from Proposition 1.

Proposition 5. *Let $k > 1$ and $c > \bar{c}^*_k$. Then,*

$$p^*_{k,c} = \phi^*_{k,c} = u^*_{k,c} = \mu^*_{k,c} = 0.$$

Let us assume in the following that $c \le \bar{c}^*_k$ holds. Lemma 2 implies that $\bar{n}_{k,c}$ is infinite in this case. Therefore, non-trivial equilibria exist for all n (Proposition 1). In the following we characterize the limit of these non-trivial equilibria for $n \to \infty$. Figure 4 illustrates.

Define

$$\lambda_{k,c}(n) = (n-1) \cdot p_{k,c}(n) \tag{20}$$

for all $n > k$. From the perspective of each player, $\lambda_{k,c}(n)$ is the expected number of other group members that will contribute in the maximal equilibrium $p_{k,c}(n)$. Appendix A.4 proves:

Lemma 3. *Let $k > 1$ and $c \le \bar{c}^*_k$. Then $\lambda^*_{k,c} = \lim_{n\to\infty} \lambda_{k,c}(n)$ is given by the unique solution to the condition $g(k-1, \lambda) = c$ that satisfies $\lambda \ge k - 1$.*

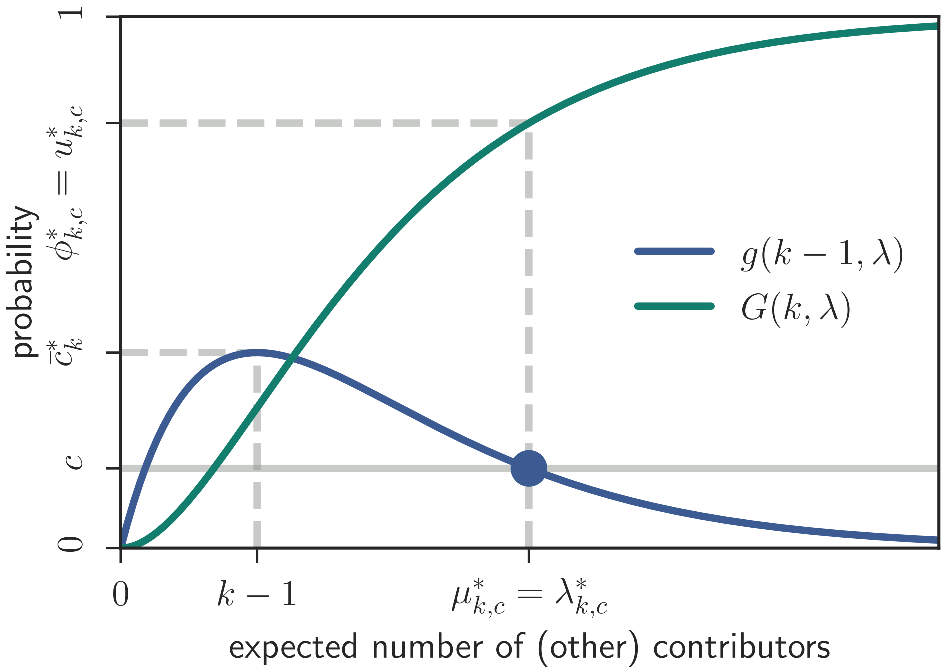

Figure 4: Illustration of Lemma 3 and Proposition 6 for $k = 2$ and $c = 0.2$. The limit of the expected number of (other) contributors $\lambda^*_{k,c} = \mu^*_{k,c}$ is given by the larger of the two solutions to the pivotality condition $g(k-1,\lambda) = c$. The limit value for both the equilibrium payoff $u^*_{k,c}$ and the probability of provision $\phi^*_{k,c}$ is the probability $G(k, \mu^*_{k,c})$ that there are at least k contributors if the number of contributors follows a Poisson distribution with expected value $\mu^*_{k,c}$.

The condition $g(k-1,\lambda) = c$ in the statement of Lemma 3 is the natural counterpart to the pivotality condition (2) when the number of other contributors follows a Poisson distribution with expected value λ. For $c < \bar{c}^*_k$, this condition has two solutions. Lemma 3 indicates that the limit value $\lambda^*_{k,c}$ of the expected number of other contributors is given by the larger of the two solutions of the pivotality condition $g(k-1,\lambda) = c$. This is analogous to Lemma 1 identifying the non-trivial maximal equilibrium as the larger of the two solutions to the pivotality condition (2). This solution satisfies $\lambda^*_{k,c} \geq k-1$ because (from Lemma 1) the inequality $\lambda_{k,c}(n) \geq k-1$ holds for all group sizes.

From the convergence of $\lambda_{k,c}(n) = (n-1) \cdot p_{k,c}(n)$ to the finite limit $\lambda^*_{k,c}$ it is immediate that $p^*_{k,c} = 0$ holds. This in turn implies that the expected number of contributors converges to the same limit as $\lambda_{k,c}(n)$, i.e., $\mu^*_{k,c} = \lambda^*_{k,c}$ holds. To prove the following result it remains to establish that both the probability of provision and the equilibrium payoff converge to the probability that there are at least k contributors if the number of contributors follows a Poisson distribution with expected value $\mu^*_{k,c}$.

This probability is $G(k, \mu^*_{k,c})$, where

$$G(x, \lambda) = \sum_{y \geq x} g(y, \lambda), \quad x = 0, 1, \dots. \tag{21}$$

Proposition 6. *Let $k > 1$ and $c \leq \bar{c}^*_k$. Then,*

$$p^*_{k,c} = 0, \mu^*_{k,c} = \lambda^*_{k,c} \text{ and } \phi^*_{k,c} = u^*_{k,c} = G(k, \mu^*_{k,c}) > 0. \tag{22}$$

Proof. Using a generalization of the classical Poisson approximation (see, for instance, Billingsley, 1995, Theorem 23.2) we have that (i) $\lambda_{k,c}(n) \to \mu^*_{k,c}$ implies $\Pi_{k,n-1}(p_{k,c}(n)) \to G(k, \mu^*_{k,c})$ and (ii) $\mu_{k,c}(n) \to \mu^*_{k,c}$ implies $\Pi_{k,n}(p_{k,c}(n)) \to G(k, \mu^*_{k,c})$. Using (6) and the second equality in (7), this proves the final two equalities in (22). Further, as $\mu^*_{k,c} > 0$ is implied by the equality $\mu^*_{k,c} = \lambda^*_{k,c}$ and Lemma 3, we have that, as asserted in (22), the inequality $G(k, \mu^*_{k,c}) > 0$ holds. □

We note that the result in Proposition 6 is a direct generalization of the limit results for the volunteer's dilemma recorded in (11). Indeed, applying (22) for $k = 1$ recovers the limit result for the volunteer's dilemma as the condition $g(0, \lambda^*_{1,c}) = c$ from Lemma 3 yields $\lambda^*_{1,c} = \ln(1/c)$ and $g(0, \lambda^*_{1,c}) = c$ implies $G(1, \lambda^*_{1,c}) = 1 - c$.

It is of interest to ask under which circumstances the expected number of contributors $\mu_{k,c}(n)$ exceeds the number of contributors k that are required for the provision of the public good.(Gradstein and Nitzan, 1990, Section 4). Lemma 3 and Proposition 6 provide an answer to this question for large group sizes. As the probability mass function $g(k, \lambda)$ of the Poisson distribution is decreasing in its parameter λ for $\lambda \geq k$ (cf. the proof of Lemma 3) the following holds: for small cost $(0 < c < g(k-1, k))$ the expected number of contributors exceeds k for sufficiently large groups, whereas for intermediate cost $(g(k-1, k) < c \leq g(k-1, k-1))$ the reverse is true.

3.7. The effect of group size on the expected number of contributors in large groups

An increase in group size has two countervailing effects on the expected number of contributors $\mu_{k,c}(n) = n \cdot p_{k,c}(n)$: on one hand, for a given probability of contribution an increase in n causes $\mu_{k,c}(n)$ to increase; on the other hand, an increase in n causes $p_{k,c}(n)$ to fall (Proposition 2). In light of Propositions 3 and 4 it is natural to conjecture that the second of these effects dominates, i.e., that the expected number of contributors is decreasing in group size. However, the following proposition shows that this is not necessarily the case. Specifically, it shows that for sufficiently large groups the comparative statics of the expected number of contributors are determined by how the participation cost c compares to a critical cost level, given by $g(k-1, k+\sqrt{k})$: for costs below this level the expected number of contributors is increasing in group size, whereas for costs above this level (but low enough for non-trivial equilibria to exist for all group-sizes) the expected number of contributors is decreasing in group size.

Proposition 7. *Let* $k > 1$.

(i) Suppose $c < g(k-1, k+\sqrt{k})$ *or, equivalently,* $\mu^*_{k,c} > k + \sqrt{k}$ *holds. Then there exists* N *such that* $\mu_{k,c}(n+1) > \mu_{k,c}(n)$ *holds for all* $n > N$.

(ii) Suppose $g(k-1, k+\sqrt{k}) < c < g(k-1, k-1)$ *or, equivalently,* $k - 1 < \mu^*_{k,c} < k + \sqrt{k}$ *holds. Then there exists* N *such that* $\mu_{k,c}(n+1) < \mu_{k,c}(n)$ *holds for all* $n > N$.

The proof of Proposition 7 is in Appendix A.5. It uses Proposition 6 only to ensure the equivalences noted in the statement of Proposition 7. In particular, the proof does *not* use Proposition 6 to approximate the binomial probabilites appearing in the pivotality condition (2) by their Poisson counterparts. Instead, our proof relies on inequalities for the probability mass function of the binomial distribution established in Anderson and Samuels (1967). This is essential, as the arguments presented in the proof of Proposition 7(iii) in Hindriks and Pancs (2002) show that using the standard Poisson approximation to the pivotality condition for finite n leads to the mistaken conclusion that for sufficiently large n the expected number of contributors is decreasing in group size irrespectively of the participation cost.[11]

4. Discussion

We have investigated group-size effects in the class of participation games without refunds introduced in Palfrey and Rosenthal (1984). We have found that for all thresholds $k > 1$ the probability of participation, the equilibrium payoff, and the probability that the public good is provided are decreasing in group size. This holds for all group sizes $n > k$ if the participation costs c are no larger than the critical value $\bar{c}^*_k$ identified in equation (18). Otherwise, these results hold for all group sizes no larger than the critical value $\bar{n}_k$ identified in equation (12) with the probability of contribution, the equilibrium payoff, and the probability of provision all being zero for group sizes exceeding $\bar{n}_k$. In line with the observation from Palfrey and Rosenthal (1984, p. 171) that "mixed strategy equilibria 'disappear' as the number of players grows large" we have found that the probability of participation converges to zero as group size goes to infinity. However, for cost $c \leq \bar{c}^*_k$ this does not imply that the equilibrium payoff and the probability of provision converge to zero. Rather, both of these converge to the same strictly positive limit that has a simple characterization in terms of the Poisson distribution.[12] We have also signed the group-size effect

[11] The arguments proving Proposition 7 are also applicable to the volunteer's dilemma. In this case we have $g(k-1, k+\sqrt{k}) = g(0, 2) = e^{-2}$, so that for participation costs c below (above) $e^{-2} \approx 0.135$ the expected number of contributors is increasing (decreasing) in group size for large n.

[12] This conclusion, as all our results, hinges on considering the limit as group size goes to infinity for a fixed threshold k. See Palfrey and Rosenthal (1984, p. 178) for a discussion of the case in which both k and n diverge to infinity.

on the expected number of contributors for large groups, finding that whether this effect is positive or negative depends on the parameters c and k. Overall, these results provide an almost complete picture of the effects of group size on the maximal symmetric equilibria in participation games without refunds.[13] In particular, the result that the probability of provision is decreasing in group size confirms the Olson conjecture for these class of games.

In their pioneering work, Palfrey and Rosenthal (1984) also consider a variant of the participation game in which contributions are refunded when the number of contributors fails to reach the necessary threshold. As shown in Palfrey and Rosenthal (1984) this participation game with refunds has a unique symmetric equilibrium $q_{k,c}(n)$. For all $c \in (0, 1)$ and $n > k$ this equilibrium satisfies $0 < q_{k,c}(n) < 1$ and the indifference condition

$$\pi_{k,n}(q_{k,c}(n)) = c \cdot \Pi_{k-1,n-1}(q_{k,c}(n)). \tag{23}$$

Despite the significant structural difference between the participation games without and with refunds, our arguments can be adapted to show that for $c \leq \bar{c}_k^*$ the Olson conjecture holds for the latter, too. We show this by establishing counterparts to Propositions 2 – 4 for the participation game with refunds in Appendix A.6.[14]

The first two of our propositions, asserting that non-trivial symmetric equilibria exist for a range of group sizes between k and some (possibly infinite) upper bound $\bar{n}_k$ (Proposition 1) and that the maximal probability of contribution that can be supported in a mixed strategy equilibrium is decreasing in group size over this range (Proposition 2), hold for a much broader class of participation games. For instance, it follows from the analysis in Peña and Nöldeke (2018) that replacing our assumption of a known threshold k with the assumption that the threshold is drawn from a distribution with support in a set of the form $\{\underline{k}, \ldots, \overline{k}\}$ as in McBride (2006) does not affect these propositions.[15] Similarly, Propositions 1 and 2 remain valid when the benefit obtained from the provision of the public good is a function of the number of contributors as in Gradstein and Nitzan (1990) or when

[13]The one missing piece is a result characterizing the group-size effect on the expected number of contributors for all (rather than only for large) group sizes. We conjecture that there are only three possibilities, namely that the expected number of contributors is (i) decreasing throughout, (ii) increasing throughout, or (iii) unimodal. Proving this conjecture is a non-trivial task as it requires to extend the analysis from Anderson and Samuels (1967) to obtain a complete characterization of the comparative statics of the binomial probability mass function when the sample size and the success probability are changed in such a way that the expected number of successes stays constant.

[14]The role of the condition $c \leq \bar{c}_k^*$ in this proof is to ensure that $q_{k,c}(n)$ exceeds the mode $(k-1)/(n-1)$ of the pivot probability $\pi_{k,n}(p)$ for all n. Whether or not the Olson conjecture also holds for the model with refunds when the participation cost exceeds $\bar{c}_k^*$ is an open question.

[15]Random thresholds arise endogenously in the complete-information voter-participation game from Palfrey and Rosenthal (1983). Group-size effects in this context have been studied in Nöldeke and Peña (2016) and Mavridis and Serena (2018).

a fixed cost of obtaining the public good is shared between all contributors once the threshold is passed. The challenge in extending the Olson conjecture to these games thus lies in identifying the conditions under which the group-size effect on the probability of contribution outweighs the countervailing effect of adding another potential contributor. The results from Weesie and Franzen (1998, Theorem 1), who show the validity of the Olson conjecture for the volunteer's dilemma with cost sharing, and Heijnen (2009), who provides sufficient conditions for the validity of the Olson conjecture in a generalized volunteer's dilemma (but also an example illustrating a failure of the Olson conjecture), are suggestive of the kind of conditions under which a counterpart to our Proposition 4 might be obtained.

Finally, we observe that Palfrey and Rosenthal (1988) have investigated group-size effects in the variants of their participation game in which agents have private information about their idiosyncratic altruistic "warm glow" benefit from contributing to the public good. The comparative statics in such a model are quite different from the ones in the complete information model that we have considered here. For instance, Palfrey and Rosenthal (1988) show that the probability of contribution first falls and then rises with an increase in group size.[16] The analysis in Palfrey and Rosenthal (1988) is complemented by the one in Goeree and Holt (2005, Proposition 4), who show that – despite the non-monotonic behavior of the probability of contribution – the Olson conjecture holds for the model from Palfrey and Rosenthal (1988) when the distribution of payoff shocks is sufficiently "noisy".

Appendix

A.1. Properties of the maximal equilibrium

Let $1 < k < n$ and let $\tilde{p} < p_{k,c}(n)$ be a symmetric equilibrium. The following argument shows that the maximal equilibrium $p_{k,c}(n)$ has a higher probability of provision and higher equilibrium payoff than $\tilde{p}$.

We note first that the same reasoning we used to obtain that the probability of provision in the maximal equilibrium is $\Pi_{k,n}(p_{k,c}(n))$ implies that the probability of provision in the equilibrium $\tilde{p}$ is $\Pi_{k,n}(\tilde{p})$. Similarly, just as the equilibrium payoff in the maximal equilibrium is given by $\Pi_{k,n-1}(p_{k,c}(n))$, the equilibrium payoff in the equilibrium $\tilde{p}$ is $\Pi_{k,n-1}(\tilde{p})$. The result then follows from observing that $\Pi_{k,m}(p)$ is increasing in p for all $1 \leq k \leq m$, i.e., for given number of trials m the binomial probability distributions for different values of the success probability p are strictly ordered in the sense of first-order stochastic dominance (see Lehmann and Romano, 2005, Chapter 3.4).

[16]In a related investigation Johnson (2002) considers group-size effects in a version of the volunteer's dilemma featuring private information both about idiosyncratic costs and benefits. Among other results he finds that – contrary to what is true in our model – equilibrium payoffs are increasing in group size.

A.2. Proof of Proposition 2

Let $P := ((k-1)/(n-1), 1)$ and $Q := [(k-1)/n, 1)$. As $\pi_{k,n}((k-1)/(n-1)) > \pi_{k,n+1}((k-1)/n)$ holds (Lemma 2), it follows from the unimodality properties of the pivot probabilities that for all $q \in Q$ there exists a unique $h(q) \in P$ such that

$$\pi_{k,n}(h(q)) = \pi_{k,n+1}(q)$$

holds. Further, the function $h : Q \to P$ thus defined is continuous (in fact, differentiable, as $\pi_{k,n+1}(q)$ is differentiable on Q and the inverse of the restriction of $\pi_{k,n}(p)$ to the interval P is differentiable by the inverse function theorem). Observing that $h((k-1)/n) > (k-1)/(n-1) > (k-1)/n$ holds, where the first inequality is from $h((k-1)/n) \in P$, and that $\pi_{k,n}(p)$ and $\pi_{k,n+1}(p)$ have no intersection in the interval P (cf., the proof of Lemma 2), we obtain $h(q) > q$ for all $q \in Q$.

The condition $1 < k < n < \bar{n}_{k,c}$ in the statement of the proposition implies $n + 1 \leq \bar{n}_{k,c}$ and therefore (Proposition 1) that non-trivial equilibria exist for group sizes n and $n+1$. Thus, Lemma 1 yields $p_{k,c}(n) \in P$, $p_{k,c}(n+1) \in Q$, and $\pi_{k,n}(p_{k,c}(n)) = \pi_{k,n+1}(p_{k,c}(n+1)) = c > 0$, so that $p_{k,c}(n) = h(p_{k,c}(n+1))$ holds. Consequently, $p_{k,c}(n) > p_{k,c}(n+1)$ follows from the inequality $h(q) > q$ established in the preceding paragraph.

A.3. Proof of Proposition 3

As in the proof of Proposition 2, let $P = ((k-1)/(n-1), 1)$, $Q = [(k-1)/n, 1)$, and let $h : Q \to P$ denote the continuous function satisfying

$$\pi_{k,n}(h(q)) = \pi_{k,n+1}(q) \tag{24}$$

for all $q \in Q$. Following reasoning analogous to the one in the proof of Proposition 2 and using (15) to rewrite the equilibrium payoffs it suffices to show

$$\Pi_{k,n-1}(h(q)) > \Pi_{k,n}(q) \quad \forall q \in Q. \tag{25}$$

Towards this end, let us define

$$\ell(x, q) = \frac{\pi_{x,n}(h(q))}{\pi_{x,n+1}(q)} \tag{26}$$

for $q \in Q$ and $x = 1, \ldots, n$. From (24) we have $\ell(k, q) = 1$ for all $q \in Q$. In the following we first argue that this implies $\ell(k+1, q) > 1$ and then, in a second step, show that this inequality implies (25). The key observation underlying these arguments is due to Klenke and Mattner (2010), who (in the proof of their Lemma 2.4) observe that, for $x = 1, \ldots, n-1$,

$$r(x, q) = \frac{\ell(x+1, q)}{\ell(x, q)} = \left(\frac{n-x}{n+1-x}\right)\left(\frac{h(q)}{1-h(q)}\right)\left(\frac{1-q}{q}\right) \tag{27}$$

is decreasing in x. The second equality in (27) follows from (1) and (26).

STEP 1: We show $\ell(k+1,q)>1$ for all $q\in Q$. Because $\ell(k,q)=1$, this is equivalent to showing that $r(k,q)>1$ holds for all $q\in Q$.

It will be useful to begin by establishing the inequality $r(k,q)>1$ for the lower endpoint of the interval Q, i.e., $q=(k-1)/n$. Substituting this value for q into equation (27) yields

$$r(k,(k-1)/n)=\left(\frac{n-k}{k-1}\right)\left(\frac{h((k-1)/n)}{1-h((k-1)/n)}\right)>1,$$

where the inequality is implied by $h((k-1)/n)>(k-1)/(n-1)$. Now suppose there exists $q\in((k-1)/n,1)$ satisfying $r(k,q)\leq 1$. Because $h:Q\to P$ is continuous in q, so is $r(k,q)$. By the intermediate value theorem for continuous functions there then exists $\hat{q}\in Q$ satisfying $r(k,\hat{q})=1$. As $r(x,\hat{q})$ is decreasing in x, this implies $r(x,\hat{q})>1$ for all x satisfying $1\leq x<k$. Consequently, $\ell(1,\hat{q})<\ell(2,\hat{q})<\ldots<\ell(k,\hat{q})=1$ holds, where the equality is from (24) and (26). Similarly, we have $r(x,\hat{q})<1$ for all x satisfying $k<x\leq n-1$, which implies $\ell(n,\hat{q})<\ell(n-1,\hat{q})<\ldots<\ell(k+1,\hat{q})=1$, where the equality is from $\ell(k,\hat{q})=1$ and $r(k,\hat{q})=1$. Hence, we have $\ell(x,\hat{q})\leq 1$ for $x=1,\ldots,n$ with strict inequality for $x\notin\{k,k+1\}$. Consequently, from (26) (and the assumption $k>1$) we have

$$\sum_{x=1}^{n}\pi_{x,n}(h(\hat{q}))<\sum_{x=1}^{n}\pi_{x,n+1}(\hat{q}). \tag{28}$$

But this is impossible: from (1) the left side of (28) is one, whereas the right side is smaller than one. Hence, no $\hat{q}$ satisfying $r(k,\hat{q})=1$ exists and we have $\ell(k+1,q)>1$ for all $q\in Q$.

STEP 2: From $\ell(k+1,q)>1$ and $\ell(k,q)=1$ we have $r(k,q)>1$. As $r(x,q)$ is decreasing in x, this implies $r(x,q)>1$ for all x satisfying $1\leq x\leq k$. Consequently, $\ell(1,q)<\ell(2,q)<\ldots<\ell(k,q)=1$ holds. Because we have assumed $k>1$ this implies

$$\sum_{x=1}^{k}\pi_{x,n}(h(q))<\sum_{x=1}^{k}\pi_{x,n+1}(q).$$

From (1) and (6) this is equivalent to (25).

A.4. Proof of Lemma 3

Upon taking logarithms in (19) it is easily verified that $g(k-1,\lambda)$ is differentiable and decreasing in λ on $[k-1,\infty)$ with $\lim_{\lambda\to\infty}g(k-1,\lambda)=0$. Hence, as asserted in the statement of the lemma, the condition $g(k-1,\lambda)=c$ has a unique solution satisfying $\lambda \geq k-1$ that we denote by $\lambda^*_{k,c}$. It remains to show that $\lambda_{k,n}(c)$ converges to this value as $n \to \infty$.

As $p_{k,c}(n)$ satisfies the pivotality condition (2) for all n, we have

$$\pi_{k,n}(\lambda_{k,c}(n)/(n-1)) = c, \quad \forall n > k.$$

From Lemma 1 we also have the inequality $\lambda_{k,c}(n) > k-1$ for all $n > k$.

Let $\epsilon > 0$ and $\overline{\lambda} = \lambda^*_{k,c} + \epsilon$. Then $g(k-1, \overline{\lambda}) < c$ holds and, by the Poisson approximation to the binomial distribution, there exists N_1 such that $\pi_{k,n}(\overline{\lambda}/(n-1)) < c$ holds for all $n > N_1$. From the unimodality properties of the pivot probability $\pi_{k,n}(p)$, we then have that $\lambda_{k,c}(n) < \overline{\lambda}$ holds for all $n > N_1$. Let $\underline{\lambda} = \lambda^*_{k,c} - \epsilon$. If $\underline{\lambda} > k-1$ holds, then, using an analogous argument to the one we used when considering $\overline{\lambda}$, there exists N_2 such that $\lambda_{k,c}(n) > \underline{\lambda}$ holds for all $n > N_2$. If $\underline{\lambda} \leq k-1$ holds, then define $N_2 = k$. We then again have that $\lambda_{k,c}(n) > \underline{\lambda}$ holds for all $n > N_2$. Letting $N = \max\{N_1, N_2\}$ we have established that for all $\epsilon > 0$ there exists N such that for all $n > N$ the inequalities $\lambda^*_{k,c} - \epsilon < \lambda_{k,c}(n) < \lambda^*_{k,c} + \epsilon$ are satisfied. Consequently, $\lambda_{k,c}(n)$ converges to $\lambda^*_{k,c}$.

A.5. Proof of Proposition 7

Proof. (i) Suppose $c < g(k-1, k+\sqrt{k})$ holds. From Proposition 6 and Lemma 3 we have that $\mu^*_{k,c} = \lim_{n\to\infty} n \cdot p_{k,c}(n)$ satisfies $g(k-1, \mu^*_{k,c}) = c$ and $\mu^*_{k,c} \geq k-1$. Because $g(k-1, \lambda)$ is decreasing in λ for $\lambda \geq k-1$ (cf. the beginning of the proof of Lemma 3) this implies $\mu^*_{k,c} > k+\sqrt{k}$. An analogous argument shows that $\mu^*_{k,c} > k+\sqrt{k}$ implies $c < g(k-1, k+\sqrt{k})$.

Suppose $\mu^*_{k,c} > k+\sqrt{k}$ holds. As $\mu_{k,c}(n) = n \cdot p_{k,c}(n)$ converges to $\mu^*_{k,c}$ there thus exists N such that $\mu_{k,c}(n) > k+\sqrt{k}$ holds for all $n > N$. Consider any such n. The first part of Theorem 3.1 in Anderson and Samuels (1967) then implies

$$\frac{\pi_{k+1,n+1}(p_{k,c}(n))}{\pi_{k+1,n+2}(p_{k,c}(n) \cdot n/(n+1))} < 1. \tag{29}$$

Simple algebra shows

$$\frac{\pi_{k+1,n+1}(p)}{\pi_{k,n}(p)} = \frac{np}{k} \text{ and } \frac{\pi_{k,n+1}(q)}{\pi_{k+1,n+2}(q)} = \frac{k}{q(n+1)}.$$

Thus,

$$\frac{\pi_{k+1,n+1}(p_{k,c}(n))}{\pi_{k+1,n+2}(n \cdot p_{k,c}(n)/(n+1))} = \frac{\pi_{k,n}(p_{k,c}(n))}{\pi_{k,n+1}(n \cdot p_{k,c}(n)/(n+1))}.$$

Hence, (29) implies

$$\pi_{k,n}(p_{k,c}(n)) < \pi_{k,n+1}(n \cdot p_{k,c}(n)/(n+1)).$$

Because $\pi_{k,n}(p_{k,c}(n)) = \pi_{k,n+1}(p_{k,c}(n+1)) = c$ holds (as both $p_{k,c}(n)$ and $p_{k,c}(n+1)$ are non-trivial), we thus have

$$\pi_{k,n+1}(p_{k,c}(n+1)) < \pi_{k,n+1}(n \cdot p_{k,c}(n)/(n+1)). \tag{30}$$

To establish the first part of the proposition, it remains to show that this inequality implies

$$p_{k,c}(n+1) > n \cdot p_{k,c}(n)/(n+1) \Leftrightarrow \mu_{k,c}(n+1) > \mu_{k,c}(n). \tag{31}$$

For $k = 1$ this is an immediate implication of (30) because $\pi_{1,n}(p)$ is strictly decreasing in p. For $k > 1$ the pivot probability $\pi_{k,n}(p)$ is strictly decreasing in p in $[(k-1)/n, 1]$ and Lemma 1 implies that $p_{k,c}(n+1) \geq (k-1)/n$ holds. Thus, a violation of the first inequality in (31) would imply that (30) is violated, too, so that the desired conclusion follows again from (30).

(ii) The equivalence $g(k-1, k+\sqrt{k}) < c < g(k-1, k-1) \Leftrightarrow k-1 < \mu^*_{k,c} < k+\sqrt{k}$ follows as the equivalence in part (i).

Suppose $k-1 < \mu^*_{k,c} < k+\sqrt{k}$ holds. Fix $\underline{\lambda}$ and $\overline{\lambda}$ such that

$$k-1 < \underline{\lambda} < \mu^*_{k,c} < \overline{\lambda} < k+\sqrt{k}$$

holds. Because $\mu_{k,c}(n) = n \cdot p_{k,c}(n)$ converges to $\mu^*_{k,c}$ there exists $N_1 > k+\sqrt{k}$ such that $\underline{\lambda} < \mu_{k,c}(n) < \overline{\lambda}$ holds for all $n > N_1$. Consider any such n. The function

$$R(\lambda, n) = \frac{\pi_{k+1,n+1}(\lambda/n)}{\pi_{k+1,n+2}(\lambda/(n+1))}$$

is unimodal in λ in $[k-1, k+\sqrt{k}]$.[17] Thus, the inequality

$$\frac{\pi_{k+1,n+1}(p_{k,c}(n))}{\pi_{k+1,n+2}(n \cdot p_{(k,c}(n)/(n+1))} \geq \min\{R(\underline{\lambda}, n), R(\overline{\lambda}, n)\}$$

holds. Observing that $k-\sqrt{k} \leq k-1$ holds, the second part of Theorem 3.1 in Anderson and Samuels (1967) implies that there exists $N_2 \geq N_1$ such that for all

[17]To verify this claim calculate

$$R(\lambda, n) = \frac{(n+1-k)}{(n+1)} \frac{(n+1)^{n+1}}{n^n} \frac{(n-\lambda)^{n-k}}{(n+1-\lambda)^{n+1-k}}$$

and observe that

$$\frac{\partial \ln(R(\lambda, n))}{\partial \lambda} = \frac{n+1-k}{n+1-\lambda} - \frac{n-k}{n-\lambda}$$

is positive for $\lambda \in [k-1, k)$ and negative for $\lambda \in (k, k+\sqrt{k}]$.

$n > N_2$ the inequality $\min\{R(\underline{\lambda}, n), R(\overline{\lambda}, n)\} > 1$ is satisfied. Hence, for all $n > N_2$ we have

$$\frac{\pi_{k+1,n+1}(p_{k,c}(n))}{\pi_{k+1,n+2}(n \cdot p_{k,c}(n)/(n+1))} > 1. \tag{32}$$

Arguments analogous to the ones showing that (29) implies part (i) of the proposition show that (32) implies $p_{k,c}(n+1) < n \cdot p_{k,c}(n)/(n+1)$, proving part (ii) of the proposition. □

A.6. The Olson conjecture for the participation game with refunds

Let $k > 1$ and suppose $c \leq \bar{c}_k^*$, so that $\bar{n}_{k,c} = \infty$ and Propositions 2 – 4 hold for all $n > k$. The following arguments show that corresponding results (and in particular, the Olson conjecture) hold for the unique symmetric equilibrium of the participation game with refunds.

We first show that the probability of contribution is decreasing in group size, i.e., that $q_{k,c}(n+1) < q_{k,c}(n)$ holds for all $n > k$. From Proposition 6 in Palfrey and Rosenthal (1984) we have $q_{k,c}(n) > p_{k,c}(n)$ for all n. Thus, using the same notation as in the proof of Proposition 2 for the sets P and Q, we have $q_{k,c}(n) \in P$ and $q_{k,c}(n+1) \in Q$ (because the equilibria $p_{k,c}(n)$ and $p_{k,c}(n+1)$ are both non-trivial by the assumption $c \leq \bar{c}_k^*$). Now suppose that $q_{k,c}(n+1) \geq q_{k,c}(n)$ holds. As $\pi_{k,n+1}(p) < \pi_{k,n}(p)$ holds on P and $\pi_{k,n+1}(p)$ is decreasing on this domain, $q_{k,c}(n+1) \geq q_{k,c}(n) \in P$ implies $\pi_{k,n+1}(q_{k,c}(n+1)) < \pi_{k,n}(q_{k,c}(n))$. We also have that $q_{k,c}(n+1) \geq q_{k,c}(n) > 0$ implies $\Pi_{k-1,n}(q_{k,c}(n+1)) > \Pi_{k-1,n-1}(q_{k,c}(n))$ as $\Pi_{k-1,n-1}(p)$ is increasing both in n and p. We thus obtain that the left side of (23) decreases when the group size is increased from n to $n+1$ whereas the right side of (23) increases, contradicting the hypothesis that $q_{k,c}(n+1)$ is the symmetric equilibrium for group size $n+1$. Hence, $q_{k,c}(n+1) < q_{k,c}(n)$ must hold.

Second, we show that the equilibrium payoff is decreasing in group size. Denoting the equilibrium payoff in the participation game without refunds as a function of group size by $v_{k,c}(n)$ we have (by the indifference condition) $v_{k,c}(n) = \Pi_{k,n-1}(q_{k,c}(n))$ for all $n > k$. Hence, our task is to show that $\Pi_{k,n}(q_{k,c}(n+1)) < \Pi_{k,n-1}(q_{k,c}(n))$ holds for all $n > k$. Towards this end, observe that (23) can be rewritten as

$$(1-c)\pi_{k,n}(q_{k,c}(n)) = c \cdot \Pi_{k,n-1}(q_{k,c}(n)). \tag{33}$$

Now suppose that $\Pi_{k,n}(q_{k,c}(n+1)) \geq \Pi_{k,n-1}(q_{k,c}(n))$ holds. From (33) we must then have $\pi_{k,n+1}(q_{k,c}(n+1)) \geq \pi_{k,n}(q_{k,c}(n)$. As $\pi_{k,n+1}(q)$ is continuous and decreasing with limit $\pi_{k,n+1}(1) = 0$ on Q, there then exists $\tilde{q} \geq q_{k,c}(n+1)$ that satisfies $\pi_{k,n+1}(\tilde{q}) = \pi_{k,n}(q_{k,c}(n)$. Because $q_{k,c}(n) \in P$ and $\tilde{q} \in Q$ hold, the same argument as in the proof of Proposition 3 implies $\Pi_{k,n}(\tilde{q}) < \Pi_{k,n-1}(q_{k,c}(n))$. But as $\Pi_{k,n}(q)$ is increasing in q, this contradicts the hypothesis $\Pi_{k,n}(q_{k,c}(n+1)) \geq \Pi_{k,n-1}(q_{k,c}(n))$.

Hence, $\Pi_{k,n}(q_{k,c}(n+1)) < \Pi_{k,n-1}(q_{k,c}(n))$ must hold, proving that $v_{k,c}(n+1) < v_{k,c}(n)$ holds.

Third, we conclude the argument by establishing that the probability of provision is decreasing in n, too.

As we are considering a non-trivial equilibrium with $q_{k,c}(n) > 0$, choosing to contribute with probability one is a best response if all other agents contribute with probability $q_{k,c}(n)$. A player choosing this strategy obtains the public good and pays the contribution cost if and only if at least $k-1$ out of the other $n-1$ group members contribute. Hence, the equilibrium payoff satisfies

$$v_{k,c}(n) = (1-c) \cdot \Pi_{k-1,n-1}(q_{k,c}(n)). \tag{34}$$

The equilibrium payoff is also given by the probability that the public good is provided if all n players contribute with probability $q_{k,c}(n)$ minus the expected cost of contribution when following this strategy. As each player contributes with probability $q_{k,c}(n)$ and in this case has to pay the cost c if and only if at least $k-1$ of the remaining $n-1$ players contribute, this means that the equilibrium payoff can also be written as

$$v_{k,c}(n) = \Pi_{k,n}(q_{k,c}(n)) - q_{k,c}(n) \cdot \Pi_{k-1,n-1}(q_{k,c}(n)) \cdot c. \tag{35}$$

Substituting from (34) into (35) to eliminate $\Pi_{k-1,n-1}(q_{k,c}(n))$ from the latter equation, we obtain

$$\Pi_{k,n}(q_{k,c}(n)) = \left[1 + \frac{q_{k,c}(n) \cdot c}{1-c}\right] v_{k,c}(n).$$

Because both $q_{k,c}(n)$ and $v_{k,c}(n)$ are decreasing in n, it follows that the probability of contribution $\Pi_{k,n}(q_{k,c}(n))$ is decreasing in n.

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