Lecture 1

EVOLUTIONARY GAME THEORY

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1 Economic theory and "as if" rationality

- The rationalistic paradigm in economics: Savage rationality [Leonard Savage: *The Foundations of Statistics*, 1954]
  - Each economic agent's behavior derived from maximization of some goal function (utility, profit), under given constraints and information

- The "as if" defence by Milton Friedman (1953): *The methodology of positive economics*
  - Firms that do not take profit-maximizing actions are selected against in the market
  - But is this claim right? Under perfect competition? Under imperfect competition?

- Evolutionary theorizing: De Mandeville, Malthus, Darwin, Maynard Smith

- Darwin: exogenous environment - “perfect competition”

- Maynard Smith: endogenous environment - “imperfect competition”

2 Game theory

- A *mathematically formalized* theory of *strategic interaction*.

- Applications abound,

— in economics

— in political science

— in biology

— in computer science

3 John Nash's two interpretations of equilibrium

Nash (Ph.D. thesis, Mathematics Department, Princeton, 1950):

1. The rationalistic (or epistemic) interpretation

2. The "mass action" (or evolutionary) interpretation

3.1 The rationalistic interpretation

1. The players interact exactly once (true even if the interaction itself is a repeated game).

2. The players are *rational* in the sense of Savage (1954)

3. Each player *knows* the game in question (strategy sets, payoffs to all players)

   However, this clearly does not imply that they will play a Nash equilibrium. Hence, assume in addition that

4. The game and all players' rationality is *common knowledge*

Example 3.1 (Coordination game)

	L	R
L	$2, 2$	$0, 0$
R	$0, 0$	$1, 1$

Example 3.2 (Matching Pennies)

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Example 3.3 *A game with a unique Nash equilibrium. This equilibrium is strict; any deviation is costly.*

	L	C	R
T	7, 0	2, 5	0, 7
M	5, 2	3, 3	5, 2
B	0, 7	2, 5	7, 0

Conclusion

The rationalistic interpretation, even under CK of the game and rationality, does not in general imply equilibrium play

[In fact, this combined hypothesis implies rationalizability.]

3.2 Nash's mass-action interpretation

1. For each *player role* in the game: a large population of identical individuals

2. The game is recurrently played, at times $t = 0, 1, 2, 3, ...$ (or at Poisson arrival times) by randomly drawn individuals, one from each *player population*

3. Individuals play *pure strategies*

4. Individuals observe *empirical samples* of earlier behaviors and outcomes and *avoid suboptimal actions*

   [According to Nash (1950), they need not know the strategy sets or payoff functions of other player roles. They do not even have to know they play a game.]

- A population-statistical distribution over the pure strategies in a player role constitute a mixed strategy

- Nash's (informal) claim: If all individuals avoid suboptimal pure strategies, and the population distribution is stationary ("stable"), then it constitutes a Nash equilibrium

  - Statistical independence across player populations $\Rightarrow$ joint probability distribution over pure-strategy profiles constitutes a mixed-strategy profile

  - Zero probability for suboptimal pure strategies $\Rightarrow$ each player-population's mixed strategy is a best reply

- Reconsider the above examples in this (informal) interpretation

Conclusion

The mass-action interpretation does not in general imply equilibrium play, but it almost does it, and holds promise

- In fact, *evolutionary game theory* provides methods and concepts to rigorously explore these promises

- The "folk theorem" of evolutionary game theory:

  - If the population process *converges* from an interior initial state, then the limit distribution is a Nash equilibrium

  - If a stationary population distribution is *stable*, then it constitutes a Nash equilibrium

– If the population process is *"convex-monotone,"* then non-rationalizable strategies have zero asymptotic probability, in all finite two-player games

4 Evolutionary game theory

- Evolutionary process =

  = mutation process + selection process

- The unit of selection: usually strategies ("strategy evolution"), sometimes utility functions ("preference evolution")

1. **Evolutionary stability**: focus on robustness to mutations

2. **Replicator dynamic**: focus on selection. [Robustness to mutations by way of dynamic stability]

3. **Stochastic stability**: both selection and mutations

5 Evolutionary stability under strategy evolution

- ESS = *evolutionarily stable strategy* [Maynard Smith and Price (1972), Maynard Smith (1973)]

  - "a strategy that 'cannot be overturned', once it has become the 'convention' in a population

Heuristically

1. A large population of individuals who are recurrently and (uniformly) randomly matched in pairs to play a finite and symmetric two-player game

2. Initially, all individuals always use the same pure or mixed strategy, x, the *incumbent* (or *resident*) strategy

3. Suddenly, a small population share $\varepsilon > 0$ switch to another pure or mixed strategy, y, the *mutant* strategy

4. If the incumbents (residents) on average *do better* than the mutants, then x is *evolutionarily stable against* y

5. x is called *evolutionarily stable* if it is evolutionarily stable against *all* mutations $y \neq x$

Domain of analysis

- *Symmetric finite two-player games in normal form*

Definition 5.1 *A* **finite** *and* **symmetric** *two-player game is any normal-form game* $G = (N, S, u)$ *with* $N = \{1, 2\}$*,* $S_1 = S_2 = S = \{1, ..., m\}$ *and* $u_2(h, k) = u_1(k, h)$ *for all* $h, k \in S$*.*

- Payoff bimatrix (A, B), where $A = (a_{hk})$, $B = (b_{hk})$

- Game symmetric iff $B = A^T$.

Example 5.1 (Prisoners' dilemma) *Payoff bimatrix:*

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$$

Symmetric since $B = A^T$.

- Write $\mathbf{\Delta}$ for $\mathbf{\Delta}(S)$, the mixed-strategy simplex:

$$\mathbf{\Delta} = \{x \in \mathbb{R}^m_+ : \sum_{i \in S} x_i = 1\}$$

- Write the payoff to any strategy $x \in \mathbf{\Delta}$, when used against any strategy $y \in \mathbf{\Delta}$ as

$$\pi(x, y) = x^T A y$$

  Note that the first argument is own strategy, second argument the other party's strategy.

- While the prisoner dilemma is symmetric, matching-pennies is not

Example 5.2 (Matching Pennies) *Payoff bimatrix:*

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here $B^T \neq A$. Not a symmetric game.

- Thus, matching pennies games fall outside the domain of evolutionary stability analysis.

- However, if player roles are randomly assigned, with equal probability for both role-allocations, then the so-defined metagame is symmetric, and evolutionary stability analysis applies to the metagame.

Example 5.3 (Coordination game) *Payoff bimatrix:*

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

$$A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A **doubly** *symmetric game:* $B = A^T = A$ *(this is an example of a 'potential game', Monderer and* **Shapley**, *1996)*

- Best replies to $x \in \Delta$:

$$\beta^*(x) = \{x^* \in \Delta : \pi(x^*, x) \geq \pi\left(x', x\right) \; \forall x' \in \Delta\}$$

- This defines a *correspondence* from Δ to itself: $\beta^* : \Delta \rightrightarrows \Delta$

  [This is distinct from the usual BR correspondence, which maps Δ^2 to Δ]

- Let

$$\Delta^{SNE} = \{x \in \Delta : x \in \beta^*(x)\}$$

- Note $x \in \Delta^{SNE} \Leftrightarrow (x, x)$ is a symmetric Nash equilibrium

Proposition 5.1 $\Delta^{SNE} \neq \varnothing$.

Proof: Application of Kakutani's Fixed-Point Theorem.

5.1 Definition of ESS

- We are now in a position to define evolutionary stability exactly:

Definition 5.2 $x \in \Delta$ *is an* **evolutionarily stable strategy (ESS)** *if for every strategy* $y \neq x$ $\exists$ $\bar{\varepsilon}_y \in (0,1)$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon}_y)$*:*

$$\pi\left[x, \varepsilon y + (1-\varepsilon)x\right] > \pi\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- "Post-entry mixture":

$$p = \varepsilon y + (1-\varepsilon)x \in \Delta$$

  a convex combination of x and y, a point on the straight line between them.

- Let $\Delta^{ESS} \subset \Delta$ denote the set of ESSs

- Note that an ESS has to be a *best* reply to itself: if $x \in \Delta^{ESS}$ then $\pi(y, x) \leq \pi(x, x)$ for all $y \in \Delta$

- Hence $\Delta^{ESS} \subset \Delta^{SNE}$

- Note also that an ESS has to be a *better* reply to its alternative best replies than they are to themselves: if $x \in \Delta^{ESS}$, $y \in \beta^*(x)$ and $y \neq x$, then $\pi(x, y) > \pi(y, y)$

Proposition 5.2 $x \in \Delta^{ESS}$ *if and only if for all* $y \neq x$*:*

$$\pi(y, x) \leq \pi(x, x)$$

and

$$\pi(y, x) = \pi(x, x) \Rightarrow \pi(y, y) < \pi(x, y)$$

5.2 Examples

5.2.1 Prisoner's dilemma

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

$$\Delta^{ESS} = \Delta^{SNE} = \{D\}$$

5.2.2 Coordination game

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

$$\Delta^{SNE} = \left\{L, R, \frac{1}{3}L + \frac{2}{3}R\right\}$$

$$\Delta^{ESS} = \{L, R\}$$

The mixed NE is *perfect* and even *proper*, but not evolutionarily stable

5.2.3 Partnership game

- Small start-up businesses with two partners, or pairs of students assigned to write an essay together

- Each partner has to choose between *work* (or "contribute") and *shirk* (or "free-ride")

  - If both choose W: *good outcome for both*

  - If one chooses W and the other S: *loss* to the first but *gain* to the second

  - If both choose S: *heavy loss to both*

- Example of payoff bimatrix:

$$\begin{array}{ccc} & W & S \\ W & 3,3 & 0,4 \\ S & 4,0 & -1,-1 \end{array}$$

- Symmetric game, but **not** a Prisoners' Dilemma: S does not dominate W: it is better to work if the other shirks

- Consider a large pool of individuals and random matching

- What do you think would happen? A tendency to work? To shirk? Any tendency at all?

- Any ESS? This is strategically equivalent to a so-called *hawk-dove game*, where work = dove and shirk = hawk

1. Unique symmetric NE: randomize uniformly, $x^* = (1/2, 1/2)$, $\Delta^{SNE} = \{x^*\}$. Hence $\Delta^{ESS} \subset \{x^*\}$

2. x^* an ESS iff

$$\pi(x^*, y) > \pi(y, y) \qquad \forall y \neq x^*$$

3. Equivalently:

$$\frac{1}{2}\left[3y_1 + 4y_1 - (1 - y_1)\right] > 3y_1^2 + 4y_1(1 - y_1) - (1 - y_1)^2$$

or

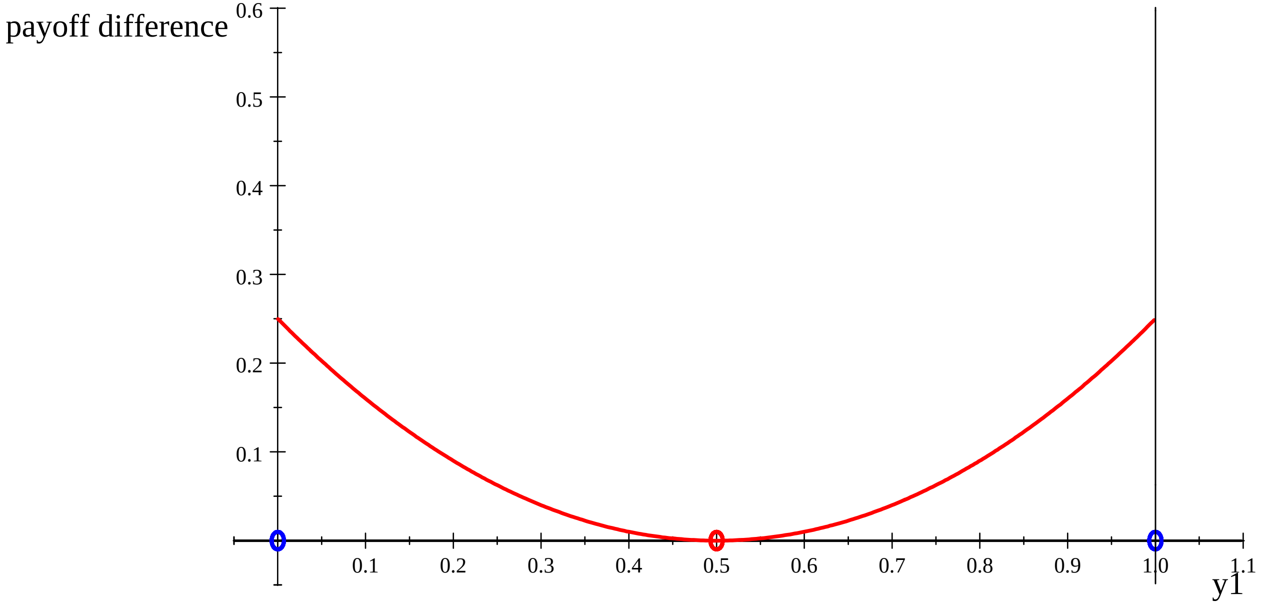
$$8y_1 - 1 > -4y_1^2 + 12y_1 - 2$$

or

$$4\left(y_1 - \frac{1}{2}\right)^2 > 0$$

4. True, hence x^* is an ESS!

- Graphical illustration of payoff difference $\pi(x^*, y) - \pi(y, y)$:



- Some games have no ESS. For instance, when all payoffs are the same. But also in more interesting games such as

Example 5.4 (Rock-scissors-paper) *Rock beats Scissors, Scissors beat Paper, and Paper beats Rock:*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Unique Nash equilibrium: $x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. *All pure strategies are best replies and do as well against themselves as* x^* *does against them.* $\Delta^{ESS} = \varnothing$.

THE END

Literature: Chapter 9 in van Damme (1991) or Chapter 2 in Weibull (1995).